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Technical Notes

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Geometric Approach to Laminar Convection

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Nomenclature

D = diameter, m
J = heat transfer rate per unit length of the pipe, W m^{-1}
k = thermal conductivity, W m^{-1} K^{-1}
L = tube cross-section perimeter, m
P = pressure gradient, Pa m^{-1}
R = radius, m
r = radial coordinate, m
S = tube cross-section, m^2
T = temperature, K
w = fluid velocity in the z-direction, m s^{-1}
x = coordinate perpendicular to the tube axis, m
y = coordinate perpendicular to the tube axis, m
z = coordinate along the tube axis, m
α = angular coordinate
β = dimensionless coefficient
Θ = torsional rigidity, m^4
μ = dynamic viscosity, m^{-1} kg s^{-1}
φ = warping function, m^2

Subscripts

H = hydraulic
m = mean
w = wall

I. Introduction

CONVECTIVE heat transfer is one of the most important topics in engineering, which is extensively discussed in standard heat transfer textbooks.[1,2] In particular, convection in internal flows, such as channel flows, is a subject of the utmost importance and can be encountered in virtually any engineering application.

In general, the solution of convective heat transfer problems cannot be obtained in a closed analytical form, with the exception of laminar flows in very simple geometries. Therefore, rather than to the study of analytical methods, much attention was dedicated to the construction and the assessment of empirical or semiempirical correlations or, alternatively, to the development of computational techniques.

Recently, laminar internal flows, such as Hagen–Poiseuille and Couette flow, have been experiencing renewed interest from researchers, which is mainly due to the growing importance of microfluidics, where low-Reynolds-number flows are quite common. Another typical feature of microchannels is that very often they have an irregular cross section, due to the difficulty of manufacturing small regular shapes with standard tools.

In this Note, the analytical approach to the solution of a steady-state convection problem for Hagen–Poiseuille flow in tubes of arbitrary cross section is reviewed, focusing on the formal analogy with continuum solid mechanics. In particular, the problem is studied from the point of view of isoperimetric inequalities, that is, of inequalities holding for domain functionals (where the domain is the tube cross-section), provided that equality is attained for some particular domain or in the limit as the domain degenerates.

This subject has been extensively studied by mathematicians,[3,4] and several applications have been found in structural or continuum solid mechanics and, more recently, in fluid mechanics.[5] The simplest isoperimetric inequality, which has been known since ancient times, states that among all plane domains of given perimeter the circle has the largest area. The study of isoperimetric inequalities in a broader sense began in 1856, when De Saint-Venant, who was investigating the torsion of elastic prisms, observed (without giving a mathematical proof, though) that of all cross sections of given area, the circle has the maximal torsional rigidity.

The concept of torsional rigidity is well known in structural mechanics, where it arises in the study of the torsion of a cylindrical beam of uniform cross section. Torsional rigidity is defined as the torque required for unit angle of twist per unit length when the elastic modulus of the material is set equal to one. Let $\varphi(x, y)$ denote the solution of the boundary value problem

$$\nabla^2\varphi = -2 \qquad \text{in } S, \qquad \varphi = 0 \qquad \text{on } \partial S \tag{1}$$

where S is a simply connected domain, and φ is a warping function[6] (which is known as the stress function in structural mechanics[7]). Then the torsional rigidity is given by

$$\Theta = 2\int_S \varphi(x, y)\,\mathrm{d}x\,\mathrm{d}y \tag{2}$$

If the heat flux through the tube wall is expressed in terms of the torsional rigidity, then all the theorems concerning this quantity can be applied straightforwardly to the heat transfer problem. This provides an analytical tool to study laminar forced convection even in channels of complex shape.

II. Analysis

Let us consider the problem of steady-state, fully developed (both thermally and hydrodynamically) Hagen–Poiseuille flow of a fluid with constant properties in a pipe of arbitrary cross section with walls at constant temperature, when there is appreciable viscous dissipation. When the z-axis is taken along the pipe length, the

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relevant equations for the velocity and the temperature fields are

$$\nabla^2 w = -\frac{P}{\mu} \tag{3}$$

$$\nabla^2 T = -\frac{\mu}{k}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \tag{4}$$

where P is the constant pressure gradient along the tube ($P > 0$). These equations are completed by the boundary conditions $w = 0$ and $T = T_w$ on ∂S. The heat transfer rate through the wall per unit length of duct is defined as

$$J = \left| \int_{\partial S} k \frac{\partial T}{\partial n} \, \mathrm{d}l \right| \tag{5}$$

Introducing Green's identities and the equations of the Hagen–Poiseuille flow yields

$$J = k\left|\int_S \nabla^2 T \, \mathrm{d}S\right| = \mu\left|\int_S \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \mathrm{d}S\right|$$
$$= \mu\left|\int_{\partial S} w \frac{\partial w}{\partial n} \, \mathrm{d}l - \int_S w \nabla^2 w \, \mathrm{d}S\right| \tag{6}$$

so that the following obvious energy balance is obtained:

$$J = P \int_S w \, \mathrm{d}S \tag{7}$$

The comparison of Eqs. (1) and (3) allows one to establish a relationship between the fluid velocity and the warping function:

$$w(x, y) = (P/2\mu)\varphi(x, y) \tag{8}$$

Expressing the heat transfer rate per unit length in terms of the warping function and recalling the definition of torsional rigidity [Eq. (2)] yields

$$J = (P^2/4\mu)\Theta \tag{9}$$

where Θ is a purely geometric factor, which does not depend on temperature levels and fluid properties, or in particular on thermal conductivity.

Because the torsional rigidity is explicitly known for several standard shapes of the tube cross section, the heat flux is known for the same domains, thanks to the energy balance equation. Table 1 reports the analytical expression of torsional rigidity for some common geometries.

The simplest case is the circular cross section. In this geometry, the temperature distribution is given by[8]

$$T = T_0 + (\mu/k)w_m^2(1 - r^4/R^4) \tag{10}$$

and substituting into Eq. (5) one obtains

$$J = \left|\int_0^{2\pi} -4\mu w_m^2 \, \mathrm{d}\alpha\right| \tag{11}$$

Solving the integral and introducing the mean velocity of the Hagen–Poiseuille flow ($w_m = PR^2/8\mu$) yields

$$J = \pi P^2 R^4/8\mu \tag{12}$$

The same result could be obtained straightforwardly from Eq. (9), substituting the torsional rigidity of the circular cross section given in Table 1.

A distinct advantage of expressing the heat rate in terms of the torsional rigidity is that the following isoperimetric inequalities can be applied, when S is a simply connected domain:

1) A cross section of circular shape yields the maximum heat transfer rate for a given area (De Saint-Venant's inequality).

2) The heat transfer rate per unit length is an increasing functional of the cross-sectional area:

$$S_1 \le S_2 \Rightarrow J_1 \le J_2 \tag{13}$$

3) The Payne–Weinberger inequality[9] obtains:

$$J \ge \left(\frac{P^2 S^2}{8\pi\mu}\right)\left[1 - \frac{2\beta^2}{1-\beta^2} - \frac{4\beta^4}{(1-\beta^2)^2}\log\beta\right] \tag{14}$$

where $\beta = 1 - 4\pi S/L^2$. For a circular cross-section, $\beta = 0$ and equality holds in Eq. (14).

4) Pólya's inequality[10] obtains:

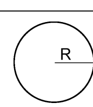
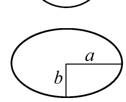
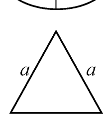
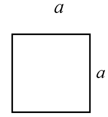
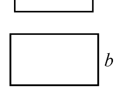
$$J > P^2 S^3/3\mu L^2 \tag{15}$$

Rigorous demonstrations of the above inequalities are not reported here, for brevity; however, they can be found in the mathematics literature.

The shape parameter β can be rewritten as $\beta = 1 - \pi D_H/L$, where $D_H = 4S/L$ is the hydraulic diameter of the channel. Thus, β indicates how the channel is hydraulically different from a circular one: for $\beta \to 0$ there is no difference, whereas the maximum difference can be observed for $\beta \to 1$.

To study the relative magnitude of Eqs. (14) and (15) one can again introduce the hydraulic diameter, so that the two inequalities

Table 1 Torsional rigidity of some common cross-sections

Shape	Dimensions	Torsional rigidity, m^4
Circle		$\dfrac{\pi R^4}{2}$
Ellipse		$\dfrac{\pi a^3 b^3}{(a^2 + b^2)}$
Triangle		$\dfrac{a^4}{15\sqrt{3}}$
Square		$2.25a^4$
Rectangle		$\dfrac{16}{3}a^3 b\left\{1 - \dfrac{192a}{\pi^5 b}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^5}\tanh\left[\dfrac{(2n-1)\pi b}{2a}\right]\right\}$

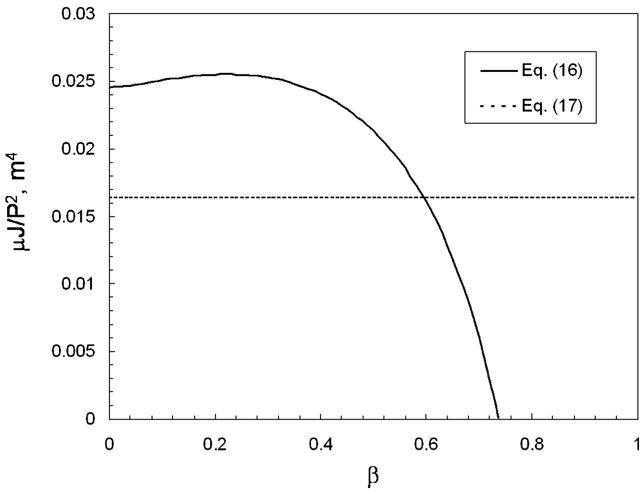


Fig. 1 Comparison between the Payne–Weinberger inequality and Polya's inequality, for $D_H = 1$.

can be rewritten respectively as

$$\frac{\mu J}{P^2} \geq \left(\frac{\pi D_H^4}{128}\right)\left[1 - \frac{2\beta^2}{1-\beta^2} - \frac{4\beta^4}{(1-\beta^2)^2}\log\beta\right] \qquad (16)$$

and

$$\mu J/P^2 \geq \pi D_H^4 \Big/ 192 \qquad (17)$$

The comparison between the two inequalities is presented graphically in Fig. 1 as a function of the shape parameter β. The curves obtained by setting the equal sign show that for $\beta < 0.6$ Eq. (16) is a more restrictive condition than Eq. (17). On the other hand, for $\beta > 0.737$ the right-hand side of Eq. (16) becomes negative, so that the inequality becomes physically meaningless.

III. Conclusions

The formal analogy with continuum solid mechanics allows one to develop a semiqualitative theory of laminar convection in tubes of arbitrary cross section, which might be useful in some cases (for instance, in microfluidics). In particular, for a fluid of constant shear viscosity driven by a fixed pressure gradient, the heat transfer rate per unit length across the wall can be expressed in terms of a purely geometric parameter, that is, the torsional rigidity of the tube cross section.

As a consequence, the heat transfer rate can be calculated in terms of overall quantities without requiring pointwise solution of the Hagen–Poiseuille problem. Furthermore, a number of mathematical theorems concerning torsional rigidity can be applied straightforward to the heat transfer problem, resulting into the definition of upper or lower bounds for the heat transfer rate through the tube wall.

Acknowledgment

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